B.sc(H) part 3 paper 5
Topic:Riemann integrrbility of continuous functions & monotonic functions
Subject: mathematics
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Theorem **Riemann integrability of continuous functions.** If a function f is continuous on $[a, b]$, then it is Riemann integrable on $[a, b]$.

Proof. Let f be continuous on $[a, b]$. Then f is bounded on $[a, b]$ and attains its *lub* and *glb* on $[a, b]$ and on every closed sub-interval of $[a, b]$. Furthermore, f is uniformly continuous on $[a, b]$. Thus given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x') - f(x'')| < \frac{\varepsilon}{b-a} \text{ provided } |x' - x''| < \delta \quad \ldots(1)$$

Choose smallest positive integer n such that $n\delta > (b-a)$. Then $\frac{b-a}{n} < \delta$. We divide the interval $[a, b]$ into n equal parts, which would imply that the length of each closed sub-interval would be less than δ. Let P be the partition of $[a, b]$ determined by this division. Let the closed sub-intervals thus determined be denoted by $\delta_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots n$. Let M_i and m_i be the *lub* and *glb* of f over δ_i. Since the values M_i and m_i are attained by f on δ_i, there exist x'_i, $x''_i \in \delta_i$ such that $f(x'_i) = M_i$, $f(x''_i) = m_i$.

But $| x'_i - x''_i | \leqslant | \delta_i | < \delta$. Therefore, it follows from (1) that

$$| f(x'_i) - f(x''_i) | < \frac{\varepsilon}{b-a}.$$

Hence $\quad M_i - m_i < \frac{\varepsilon}{b-a}$ for $i = 1, 2, \ldots, n$.

Now $\quad U(P) - L(P)$

$$= \sum_{i=1}^{n} (M_i - m_i) \, | \delta_i |$$

$$< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a}\right) | \delta_i |$$

$$= \frac{\varepsilon}{b-a} \sum_{i=1}^{n} | \delta_i | = \frac{\varepsilon}{b-a} (b-a) = \varepsilon.$$

We have thus shown that given any $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(P) - L(P) < \varepsilon.$$

Hence f is Riemann integrable on $[a, b]$. Thus every continuous function on $[a, b]$ is Riemann integrable on $[a, b]$.

Remark. The converse of the above theorem is not necessarily true. As an example to show this consider the following :

Example of a function which is Riemann integrable over a bounded closed interval but not continuous over it.

Let $f : [0, 2] \rightarrow R$ be defined by

$$f(x) = 2 \text{ for } x \in [0, 1]$$
$$f(x) = 3 \text{ for } x \in [1, 2]$$

It can be easily seen that for any $\varepsilon > 0$ there is a partition P of [0, 2] such that it contains a sub-interval containing 1 and whose length is less than ε. Then it would follow that $U(P) - L(P) < \varepsilon$. Hence f is Riemann integrable. However, f is discontinuous at 1.

Riemann integrability of monotonic function

Theorem. Every bounded monotonic function $f : [a, b] \to \mathbf{R}$ is Riemann integrable on $[a, b]$.

Proof. It suffices to prove the result for the case of a bounded monotonically increasing function since the argument would be similar for a bounded monotonically decreasing function.

In case $f(a) = f(b)$ then f would be constant, and therefore Riemann integrable.

Let us now assume that $f(a) < f(b)$. Given $\varepsilon > 0$, we construct a partition P of $[a, b]$ such that the length of each sub-interval is less than $\delta = \frac{\varepsilon}{f(b) - f(a)}$. If $a = x_0 < x_1 < x_2 < \ldots\ldots < x_n = b$ be the partitioning points and $\delta_i = [x_{i-1}, x_i]$ then

$$U(P) = \sum_{i=1}^{n} M_i \mid \delta_i \mid = \sum_{i=1}^{n} f(x_i) \mid \delta_i \mid$$

and $$L(P) = \sum_{i=1}^{n} m_i \mid \delta_i \mid = \sum_{i=1}^{n} f(x_{i-1}) \mid \delta_i) \mid$$

Hence $$U(P) - L(P) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \mid \delta_i \mid$$

But $\mid \delta_i \mid < \frac{\varepsilon}{f(b) - f(a)}$ for each $i = 1, 2, \ldots, n$.

Thus $U(P)-L(P)<\frac{\varepsilon}{f(b)-(a)}\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]$

$$=\frac{\varepsilon}{f(b)-f(a)}[f(b)-f(a)]=\varepsilon.$$

Thus given $\varepsilon>0$, there exists a partition P such that

$$U(P)-L(P)<\varepsilon.$$

Therefore f is Riemann integrable.